Educaţia Matematică Vol. 4, Nr. 1 (2008), 65 - 75

On a diophantine equation of $a^x + b^y = z^2$ type [1]

Codruţa Elena Pumnea, Amada Dorina Nicoară

Abstract

In this paper we study in natural numbers some diophantine equation of $a^x + b^y = z^2$ type.

2000 Mathematical Subject Classification: 11D61

In this note we study diophantine equation of $a^x + b^y = z^2$ type, where $a, b, c \in \mathbb{N}^*, a, b \geq 2, a \neq b$. In their study we use the method of modular arithmetics.

1. Equation $2^x + 7^y = z^2$. We have the following result:

Proposition 1.1. *The diophantine equation* $2^x + 7^y = z^2$ *has exactly three solutions*

$$(x, y, z) \in \{(3, 0, 3), (5, 2, 9), (1, 1, 2)\}.$$

Proof. We consider several cases.

Case 1.1. For $x = 0$, then we have the diophantine equation

$$7^y = z^2 - 1$$

or

$$(z - 1)(z + 1) = 7^y,$$

[1] *Received 28 April, 2008*

Accepted for publication (in revised form) 20 May, 2008

where $z-1=7^u$ and $z+1=7^{y-u}, y>2u, u\in\mathbb{N}^*$.

From here, we obtain:

$$7^{y-u}-7^u=2$$

or

$$7^u(7^{y-2u}-1)=2,$$

where $u=0$ and $7^y=3$, which is impossible.

Case 1.2. If $y=0$, then we have the diophantine equation

$$z^2-1=2^x$$

or

$$(z-1)(z+1)=2^x,$$

where $z-1=2^v$ and $z+1=2^{x-v}, x>2v, v\in\mathbb{N}^*$.
From here, we obtain

$$2^{x-v}-2^v=2$$

or

$$2^v(2^{x-2v}-1)=2.$$

where $v=1$ and $2^{x-2}=2$, that is $v=1$ and $x=3$.
Therefore $x=3,\ y=0,\ z=3$.

Case 1.3. x even. Now, we consider $x=2k, k\in\mathbb{N}^*$ we have

$$z^2-2^{2k}=7^y$$

or

$$(z-2^k)(z+2^k)=7^y,$$

where $z-2^k=7^u$ and $z+2^k=7^{y-u}, y>2u, y\in\mathbb{N}^*$.

From here, we obtain

$$7^{y-u}-7^u=2^{k+1}$$

or

$$7^u(7^{y-2u}-1)=2^{k+1},$$

which implies $u = 0$ and $7^y - 1 = 2^{k+1}$.

If $y \geq 1$ we have

$$(7-1)t = 2^{k+1}, t \in \mathbb{N}^*$$

or

$$6t = 2^{k+1},$$

it results that 3 divides 2^{k+1}, which is impossible.

Case 1.4. y even. We consider $y = 2k, k \in \mathbb{N}^*$ we have

$$z^2 - 7^{2k} = 2^x$$

or

$$(z - 7^k)(z + 7^k) = 2^x,$$

we have $z - 7^k = 2^v$ and $z + 7^k = 2^{x-v}, x > 2v, v \in \mathbb{N}^*$.

From here, we obtain

$$2^{x-v} - 2^v = 2 \cdot 7^k$$

or

$$2^v(2^{x-2v} - 1) = 2 \cdot 7^k,$$

which implies $v = 1$ and

$$2^{x-2} - 1 = 7^k.$$

As $7^k \equiv 0(\bmod 7), k \in \mathbb{N}^*$ and $2^{x-2} - 1 \equiv 0(\bmod 7)$, only if $x - 2 = 3p, p \in \mathbb{N}^*$. Then:

(1) $$2^{3p} - 1 = 7^k,$$

it results $(7+1)^p - 1 = 7^k$.

Using the Newton‘s binomial it results

$$7^2t + 7p = 7^k, t \in \mathbb{N}^*,$$

or

(2) $$7t + p = 7^{k+1}, t \in \mathbb{N}^*.$$

If $k = 1$, then we have $p = 1$ and we have the solution

$$x = 5,\ y = 2,\ z = 9.$$

For $k \geq 2$, from (2) results $p = 7s, s \in \mathbb{N}^*$. Then from (1) we have

$$2^{21s} - 1 = 7^k$$

or

$$(2^7)^{3s} - 1 = 7^k$$

or

$$(2^7 - 1)q = 7^k, q \in \mathbb{N}^*,$$

from where we obtain $127q = 7^k$, which is impossible because 127 it is not divisible by 7.

Case 1.5. $x,\ y$ odd. Because z is odd, then we have

$$z = 2p + 1,$$

and it results $z^2 = 4p^2 + 4p + 1 = 4p(4p + 1) + 1 \equiv 1(\bmod 8)$.
For $x \geq 3$ and odd, we have

$$2^x \equiv 0(\bmod 8),$$

and for y odd we have $7^{2k+1} = 7^{2k} \cdot 7 \equiv 7(\bmod 8)$.

From here, we obtain

$$2^x + 7^y \equiv 7(\bmod 8),$$

which is impossible, because $z^2 \equiv 1(\bmod 8)$.

If $y = 1$ results $z = 3$ therefore $x = 1,\ y = 1,\ z = 3$.

If $x = 1$ and $y \geq 3$ odd, we have

$$2 + 7^y = z^2$$

or

$$z^2 - 7^y = 2,$$

where $z = 2p + 1, p \geq 2$ and $y = 2q + 1, q \geq 1$, because for $z = 1$ the equation has no solution.

Because $7^y = (6 + 1)^y \equiv 1(\bmod 6)$, there are three cases:

$$z = 6p + 1,\ 6p + 3,\ 6p + 5.$$

If $z = 6p + 1$ we have $z^2 \equiv 1(\bmod 6)$, so $z^2 - 7^y \equiv 0(\bmod 6)$ which is a contradiction with $2 \equiv 2(\bmod 6)$.
If $z = 6p + 5$ we have $z^2 \equiv 1(\bmod 6)$, so $z^2 - 7^y \equiv 0(\bmod 6)$ which is a contradiction with $2 \equiv 2(\bmod 6)$.
If $z = 6p+3$ we have $z^2 = 36p^2 + 36p + 9 = 36p(p+1) + 9 \equiv 0(\bmod 9)$, it results

$$z^2 - 7^y \equiv 1, 4, 7(\bmod 9),$$

which is a contradiction with $2 \equiv 2(\bmod 9)$.
In concluding the diophantine equation (1) has three solutions $(x, y, z) \in \{(3, 0, 3), (5, 2, 9), (1, 1, 2)\}$.

2. Equation $2^x + 11^y = z^2$. We have the following result:

Proposition 2.1. *The diophantine equation* $2^x + 11^y = z^2$ *has exactly one solution* $(x, y, z) = (3, 0, 3)$.

Proof. We consider several cases:

Case 2.1. If $x = 0$, then we have the diophantine equation

$$11^y = z^2 - 1$$

or

$$(z - 1)(z + 1) = 11^y,$$

where $z - 1 = 11^u$ and $z + 1 = 11^{y-u}, y > 2u, u \in \mathbb{N}^*$.

From here, we obtain:

$$11^{y-u} - 11^u = 2$$

or

$$11^u(11^{y-2u} - 1) = 2,$$

where $u = 0$ and $11^y = 3$, which is impossible.

Case 2.2. If $y = 0$, then we have the diophantine equation

$$z^2 - 1 = 2^x$$

or

$$(z-1)(z+1) = 2^x,$$

where $z - 1 = 2^v$ and $z + 1 = 2^{x-v}, x > 2v, v \in \mathbb{N}^*$.

From here, we obtain

$$2^{x-v} - 2^v = 2$$

or

$$2^v(2^{x-2v} - 1) = 2.$$

where $v = 1$ and $2^{x-2} = 2$, that is $v = 1$ and $x = 3$.

Therefore $x = 3,\ y = 0,\ z = 3$.

Case 2.3. x even. Now, we consider $x = 2k, k \in \mathbb{N}^*$ we have

$$z^2 - 2^{2k} = 11^y$$

or

$$(z - 2^k)(z + 2^k) = 11^y,$$

where $z - 2^k = 11^u$ and $z + 2^k = 11^{y-u}, y > 2u, y \in \mathbb{N}^*$.

From here, we obtain

$$11^{y-u} - 11^u = 2^{k+1}$$

or

$$11^u(11^{y-2u} - 1) = 2^{k+1},$$

which implies $u = 0$ and

$$11^y - 1 = 2^{k+1}.$$

If $y \geq 1$ we have

$$10t = 2^{k+1},\ t \in \mathbb{N}^*$$

where it results that 5 divides 2^{k+1}, which is impossible.

Case 2.4. y even. We consider $y = 2k, k \in \mathbb{N}^*$ we have

$$z^2 - 11^{2k} = 2^x$$

or

$$(z - 11^k)(z + 11^k) = 2^x,$$

we have $z - 11^k = 2^v$ and $z + 11^k = 2^{x-v}, x > 2v, v \in \mathbb{N}^*$.

From here, we obtain

$$2^{x-v} - 2^v = 2 \cdot 11^k$$

or

$$2^v(2^{x-2v} - 1) = 2 \cdot 11^k,$$

which implies $v = 1$ and $2^{x-2} - 1 = 11^k$. Using modulo 11 we have

$$x - 2 = 10p,\ p \in \mathbb{N}^*$$

or

$$2^{10p+2} - 1 = 11^k$$

or

$$4^{5p+1} - 1 = 11^k,$$

from where we obtain $3t = 11^k$, where $t \in \mathbb{N}^*$, so 3 divides 11, which is impossible.

Case 2.5. $x,\ y$ odd. Because z is odd, then we have

$$z = 2p + 1,$$

it results $z^2 = 4p^2 + 4p + 1 = 4p(4p + 1) + 1 \equiv 1(\bmod 8)$.

If $x \geq 3$ and odd, we have $2^x \equiv 0(\bmod 8)$, and if y odd we have $11^{2k+1} = 11^{2k} \cdot 11 \equiv 3(\bmod 8)$.

From here, we obtain

$$2^x + 11^y \equiv 3(\bmod 8),$$

which is impossible, because $z^2 \equiv 1(\mod 8)$.
If $x = 1$ and y odd, we have $2+11^y \equiv 5(\mod 8)$ which is impossible because $z^2 \equiv 1(\mod 8)$.
In concluding the diophantine equation (2) has a solution $(x, y, z) = (3, 0, 3)$.
3. Equation $2^x + 13^y = z^2$. We have the following result:
Proposition 3.1. *The diophantine equation* $2^x + 13^y = z^2$ *has exactly one solution* $(x, y, z) = (3, 0, 3)$.
Proof. We consider several cases.
Case 3.1. If $x = 0$, then we have the diophantine equation

$$13^y = z^2 - 1$$

or

$$(z - 1)(z + 1) = 13^y,$$

where $z - 1 = 13^u$ and $z + 1 = 13^{y-u}, y > 2u, u \in \mathbb{N}^*$.

For here, we obtain

$$13^{y-u} - 13^u = 2$$

or

$$13^u(13^{y-2u} - 1) = 2,$$

where $u = 0$ and $13^y = 3$, which is impossible.
Case 3.2. If $y = 0$, then we have the diophantine equation

$$z^2 - 1 = 2^x$$

or

$$(z - 1)(z + 1) = 2^x,$$

where $z - 1 = 2^v$ and $z + 1 = 2^{x-v}, x > 2v, v \in \mathbb{N}^*$.

From here, we obtain

$$2^{x-v} - 2^v = 2$$

or

$$2^v(2^{x-2v} - 1) = 2.$$

where $v = 1$ and $2^{x-2} = 2$, that is $v = 1$ and $x = 3$.
Therefore $x = 3,\ y = 0,\ z = 3$.
Case 3.3. x even. We consider $x = 2k, k \in \mathbb{N}^*$ we have

$$z^2 - 2^{2k} = 13^y$$

or

$$(z - 2^k)(z + 2^k) = 13^y,$$

we have $z - 2^k = 13^y$ and $z + 2^k = 13^{y-u}, y > 2u, y \in \mathbb{N}^*$.

From here, we obtain

$$13^{y-u} - 13^u = 2^{k+1}$$

or

$$13^u(13^{y-2u} - 1) = 2^{k+1},$$

where $u = 0$ and $13^y - 1 = 2^{k+1}$. If $y \geq 1$ we have

$$(13 - 1)t = 2^{k+1}, t \in \mathbb{N}^*$$

or

$$12t = 2^{k+1},$$

from where it results that 3 divides 2^{k+1}, which is impossible.
Case 3.4. y even. We consider $y = 2k, k \in \mathbb{N}^*$ we obtain

$$z^2 - 13^{2k} = 2^x$$

or

$$(z - 13^k)(z + 13^k) = 2^x,$$

where $z - 13^k = 2^x$ and $z + 13^k = 2^{x-v}, x > 2v, v \in \mathbb{N}^*$.

From here, we obtain

$$2^{x-v} - 2^v = 2 \cdot 13^k$$

or

$$2^v(2^{x-2v}-1)=2\cdot 13^k,$$

where $v=1$ and $2^{x-2}-1=13^k$,
and $x-2=12p,$ where $p\in\mathbb{N}^*$ or

$$x=12p+2.$$

We have,

$$2^{x-2}-1=(2^2)^{6p+1}-1=3t, t\in\mathbb{N}^*,$$

3 divides 13, which is impossible.
Case 3.5. $x,\ y$ odd. Because z is odd, then we have

$$z=2p+1,$$

it results $z^2=4p^2+4p+1=4p(4p+1)+1\equiv 1(\bmod 8)$.

If $x\geq 3$ we have $2^x\equiv 0(\bmod 8)$, and if y odd we have $13^{2k+1}=13^{2k}\cdot 13\equiv 3(\bmod 8)$.

From here, we obtain

$$2^x+13^y\equiv 5(\bmod 8),$$

which is a contradiction, because $z^2\equiv 1(\bmod 8)$.

If $x=1$ and y odd, we have

$$2+13^y\equiv 7(\bmod 8)$$

which is impossible because $z^2\equiv 1(\bmod 8)$.

In concluding the diophantine equation (3) has only the solution

$$(x,y,z)=(3,0,3).$$

References

[1] D. Acu, *On a Diophantine equation*, General Mathematics, Vol15, Nr.4, 2007;

[2] D. Acu, *Aritmetică şi teoria numerelor*, Editura Universitatea "Lucian Blaga" din Sibiu, 1999;

[3] I. Cucurezeanu, *Ecuaţii în numere întregi*, Editura Aramis, 2006.

Departament of Mathematics,
Faculty of Sciences,
University "Lucian Blaga" of Sibiu,
Dr. Ion Ratiu 5-7, Sibiu, 550012, Romania