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Characteristic polynomial of A and Faddeev's method for A^{-1}[1]

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Abstract

We explain that, the Leverrier-Takeno's procedure for to construct the characteristic equation of an arbitrary matrix A leads, via Cayley-Hamilton theorem, to Faddeev's algorithm for A^{-1}

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1 Introduction

For any matrix $A_{n\times n} = (a^i_j)$ its characteristic polynomial:

$$\lambda^n + \alpha_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n = 0 \tag{1}$$

can be obtained, through several methods [1-5], directly from the condition $\det(a^i_j - \lambda\delta^i_j) = 0$. The technique of Leverrier-Takeno [1,6-10], presented in Sec. 2, is a simple and interesting method for to construct (1) based it in the traces of the powers A^r, $r = 1, \ldots, n$. On the other hand, it is very known that any matrix satisfies its characteristic equation:

$$A^n + a_1A^{n-1} + \ldots + a_{n-1}A + a_nI = 0 \tag{2}$$

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which represents the Cayley-Hamilton theorem. If A is nonsingular ($\det A \neq 0$), then (2) permits to deduce the corresponding inverse matrix:

$$A^{-1} = -\frac{1}{a_n}(A^{n-1} + a_1 A^{n-2} + \ldots + a_{n-1} I), \tag{3}$$

in this case we have $a_n \neq 0$ because $a_n = (-1)^N \det A$.

Fadeev [11,12] proposed an algorithm, explained in Sec.3, for to determine A^{-1} in terms of the A^r and their traces. Here we shall exhibit that (3) coincides with the Faddev's result if we employ the Leverrier-Takeno's expressions for the coefficients a_j. In other words, the procedure of Faddeev is equivalent to Cayley-Hamilton equation.

2 Leverrier-Takeno method

If we define quantities:

$$a_0 = 1, \qquad S_r = tr A^r, \qquad r = 1, 2, \ldots, n \tag{4}$$

then the technique of Leverrier-Takeno [1,6-10] leads to (1) in where the a_i are determined with the following recurrent formulae:

$$r a_r + s_1 a_{r-1} + s_2 a_{r-2} + \ldots + s_{r-1} a_1 + s_r = 0, \qquad r = 1, 2, \ldots, n \tag{5}$$

therefore

$$\begin{gathered} a_1 = -s_1, \qquad 2! a_2 = (s_1)^2 - s_2, \qquad 3! a_3 = -(s_1)^3 + 3 s_1 s_2 - 2 s_3, \\ 4! a_4 = (s_1)^4 - 6 (s_1)^2 + 8 s_1 s_3 + 3 (s_2)^2 - 6 s_4, \quad etc. \end{gathered} \tag{6}$$

in particular $\det A = (-1)^n a_n$, that is the determinant of any square matrix only depends of the traces s_r, thus it is evident than A and its transpose have the same determinant.

This method based in the relations (4) and has applications in numerical analysis, but it also is very useful in theoretical work on several topics of general relativity as the embedding of Riemannian spaces [13-15], algebraic studies of the Ricci tensor [16-23], alternative gravitational theories [24], etc. because there it is necessary to determine the characteristic polynomial of a second order tensor. Besides, the Leverrier-Takeno procedure finds utility in the study of the motion of a classical charged particles into a uniform electromagnetic field [25-29].

3 Faddeev Method

The algorithm proposed by Faddeev [11,12] for to obtain A^{-1} is a sequence of algebraic operations on the powers A^r and their traces, in fact, his method is given by the following instructions:

(7)

$$\begin{array}{lll} A_1 = A, & q_1 = trA_1, & B_1 = A_1 - q_1 I, \\ A_2 = B_1 A, & q_2 = \frac{1}{2} trA_2, & B_2 = A_2 - q_2 I, \\ \vdots & \vdots & \vdots \\ A_{n-1} = B_{n-2} A, & q_{n-1} = \frac{1}{n-1} trA_{n-1} & B_{n-1} = A_{n-1} - q_{n-1} I, \\ A_n = B_{n-1} & q_n = \frac{1}{n} trA_n & \end{array}$$

then

(8) $$A^{-1} = \frac{1}{q_n} B_{n-1}$$

As an example, if we apply (7) tot the case $n = 4$, then it is easy to see that the corresponding q_r reproduce (6) resulting $q_j = -a_j$, and besides the expression (8) is identical to (3) from Cayley-Hamilton identity. By mathematical induction really is immediate to prove that (7) and (8) are equivalent to (3), (4) and (5), showing thus that the Faddeev technique has its origin in the Leverrier-Takeno method plus Cayley-Hamilton theorem.

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