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A normed space generated by a real function [1]

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Abstract

We consider a real function f so that $\lim_{x\to\infty} f(x) = 0$ and the normed space generated by a family of functions g for that $\lim_{x\to\infty} g(x)f(x)$ are finite.

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For $a \in \mathbb{R}$ we consider a fixed real function $f : (a, \infty) \to \mathbb{R}$ which

$$\lim_{x\to\infty} f(x) = 0$$

and respectively, the space

$$\mathcal{F}(f) = \{g \ / \ g : (a, \infty) \to \mathbb{R},\ \lim_{x\to\infty} g(x)f(x) \text{ are finite}\}.$$

Theorem 1. *The space $\mathcal{F}(f)$ is a real linear space relative to the usual addition of functions, respectively to the scalar multiplication.*

Proof. If $g, h \in \mathcal{F}(f)$ then

$$(g(x) + h(x))f(x)) = g(x)f(x) + h(x)f(x)$$

has a finitely limit to ∞ and $g+h \in \mathcal{F}(f)$. Similarly, if $g \in \mathcal{F}(f)$ and $\lambda \in \mathbb{R}$ then $(\lambda g)(x) = \lambda g(x)$ has a finitely limit to ∞ and $\lambda g \in \mathcal{F}(f)$.

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Theorem 2. *The binary relation defined on $\mathcal{F}(f)$ by $g \sim h$ if and only if*

$$\lim_{x\to\infty} g(x)f(x) = \lim_{x\to\infty} h(x)f(x)$$

is an equivalence relation.

The proof is immediately.

Theorem 3. *The quotient space $\mathcal{F}(f)/\sim$, denoted by $F(f)$ is also a real linear space.*

Proof. We will show that the equivalence relation $\sim$ is compatible with the linear structure of $\mathcal{F}(f)$.

Let $g \sim g'$ and $h \sim h'$. Then,

$$\lim_{x\to\infty}(g(x)+h(x))f(x) = \lim_{x\to\infty}(g(x)f(x)+h(x)f(x)) =$$

$$\lim_{x\to\infty} g(x)f(x) + \lim_{x\to\infty} h(x)f(x) = \lim_{x\to\infty} g'(x)f(x) + \lim_{x\to\infty} h'(x)f(x) =$$

$$\lim_{x\to\infty}(g'(x)f(x)+h'(x)f(x)) = \lim_{x\to\infty}(g'(x)+h'(x))f(x)$$

and, consequently, $g+h \sim g'+h'$.

Let $g \sim g'$ and $\lambda \in \mathbb{R}$. Then

$$\lim_{x\to\infty}(\lambda g(x))f(x) = \lim_{x\to\infty}\lambda(g(x)f(x)) = \lambda \lim_{x\to\infty} g(x)f(x) =$$

$$\lambda \lim_{x\to\infty} g'(x)f(x) = \lim_{x\to\infty}\lambda(g'(x)f(x)) = \lim_{x\to\infty}(\lambda g'(x))f(x),$$

and, finally, $\lambda g \sim \lambda g'$.

Theorem 4. *The real valued function defined on $F(f)$ by*

$$\hat{g} \mapsto \|\hat{g}\| = \lim_{x\to\infty}|g(x)f(x)|,$$

where $g \in \hat{g}$, is a norm on $F(f)$. Thus, $F(f)$ becomes a normed linear space.

Proof. First, let us show that the value of the function is independent of the particular representative chosen in the class $\hat{g}$ (the function is well-defined). Let

$$g',\ g'' \in \hat{g},$$

hence

$$\lim_{x\to\infty} g'(x)f(x) = \lim_{x\to\infty} g''(x)f(x).$$

It results that the functions $|g'(x)f(x)|,\ |g''(x)f(x)|$ have also the same limit to ∞.

We have

$$\|\hat{0}\| = \lim_{x\to\infty} |0f(x)| = 0$$

and conversely, if

$$\|\hat{g}\| = 0, \text{ then } \lim_{x\to\infty} |g(x)f(x)| = 0,$$

where $g \in \hat{g}$. Then also

$$\lim_{x\to\infty} g(x)f(x) = 0,$$

therefore $g \in \hat{0}$, this means $\hat{g} = \hat{0}$.

Let $\alpha \in \mathbb{R}$ and $\hat{g} \in F(f))$. If $g \in \hat{g}$ then $\alpha g \in \alpha\hat{g}$ and we can write

$$\|\alpha\hat{g}\| = \lim_{x\to\infty} |(\alpha g(x))f(x)| = |\alpha| \lim_{x\to\infty} |g(x)f(x)| = |\alpha|\|\hat{g}\|.$$

Let $\hat{g}, \hat{h} \in F(f)$ and $g \in \hat{g},\ h \in \hat{h}$. Obviously,

$$g + h \in \widehat{g+h} = \hat{g} + \hat{h}$$

and, consequently,

$$\|\hat{g} + \hat{h}\| = \lim_{x\to\infty} |(g(x) + h(x))f(x)| =$$

$$\lim_{x\to\infty} |g(x)f(x) + h(x)f(x)| \leq \lim_{x\to\infty} (|g(x)f(x)| + |h(x)f(x)|) =$$

$$\lim_{x\to\infty} |g(x)f(x)| + \lim_{x\to\infty} |h(x)f(x)| = \|\hat{g}\| + \|\hat{h}\|.$$

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