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A remark on the first kind of improper integrations

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Abstract

The object of the present note is to show a new approach for the first kind of improper integration.

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1 Improper integration I_1

Let us consider the functions $f(x)$ which are continuous in the open interval (a, b) and discontinuous at the points $x = a$ and $x = b$. For such functions $f(x)$, the improper integration

$$I_1 = \int_a^b f(x)dx$$

is given by

$$I_1 = \lim_{\substack{\varepsilon \to +0 \\ \delta \to +0}} \int_{a+\varepsilon}^{b-\delta} f(x)dx$$

.

For this improper integration I_1, we consider, for c such that $a < c < b$,

$$I_1 = \lim_{\varepsilon \to +0} \int_a^c f(x+\varepsilon)dx + \lim_{\delta \to +0} \int_a^b f(x-\delta)dx.$$

Example 1. *Let us consider the improper integration*

$$I_1 = \int_{-1}^1 \frac{1}{\sqrt{1-x^2}}dx.$$

Since the function $f(x) = \dfrac{1}{\sqrt{1-x^2}}$ *is not continuous at* $x = -1$ *and* $x = 1$*, we have, for* c *such that* $-1 < c < 1$*,*

$$I_1 = \lim_{\varepsilon \to +0} \int_{-1}^c \frac{1}{\sqrt{1-(x+\varepsilon)^2}}dx + \lim_{\delta \to +0} \int_c^1 \frac{1}{\sqrt{1-(x-\delta)^2}}dx =$$

$$= \lim_{\varepsilon \to +0} \int_{-1+\varepsilon}^{c+\varepsilon} \frac{1}{\sqrt{1-t^2}} + \lim_{\delta \to +0} \int_{c-\delta}^{1-\delta} \frac{1}{\sqrt{1-u^2}}du =$$

$$\lim_{\varepsilon \to +0} \left[\sin^{-1} t\right]_{-1+\varepsilon}^{c+\varepsilon} + \lim_{\delta \to +0} \left[\sin^{-1} u\right]_{c-\delta}^{1-\delta} =$$

$$\sin^{-1} c - \sin^{-1}(-1) + \sin^{-1}(1) - \sin^{-1} c = \pi.$$

2 Improper integration I_2

Next, if a function $f(x)$ is continuous in $[a, b]$ except for a point $x = c$ such that $a < c < b$, then the improper integration

$$I_2 = \int_a^b f(x)dx$$

is defined by

$$I_2 = \lim_{\varepsilon \to +0} \int_a^{c-\varepsilon} f(x)dx + \lim_{\delta \to +0} \int_{c+\delta}^b f(x)dx.$$

For this improper integration, we consider

$$I_2 = \lim_{\varepsilon \to +0} \int_a^c f(x-\varepsilon)dx + \lim_{\delta \to +0} \int_c^b f(x+\delta)dx.$$

Example 2. *Let us consider the improper integration*

$$I_2 = \int_{3/2}^{5/2} x[x]dx,$$

where [ ] means the Gauss symbol. It is clear that the function $f(x) = x[x]$ *is continuous in* $\left[\frac{3}{2}, \frac{5}{2}\right]$ *except for a point* $x = 2$*. Therefore, we have*

$$I_2 = \lim_{\varepsilon \to +0} \int_{3/2}^{2} (x-\varepsilon)[x-\varepsilon]dx + \lim_{\delta \to +0} \int_{2}^{5/2} (x+\delta)[x+\delta]dx.$$

Note that $[x-\varepsilon] = 1$ *for* $\frac{3}{2} \leqq x \leqq 2$*, and* $[x+\delta] = 2$ *for* $2 \leqq x \leqq \frac{5}{2}$*. This gives us that*

$$\begin{aligned} I_2 &= \lim_{\varepsilon \to +0} \int_{3/2}^{2} (x-\varepsilon)dx + \lim_{\delta \to +0} \int_{2}^{5/2} 2(x+\delta)dx \\ &= \lim_{\varepsilon \to +0} \int_{(3/2)-\varepsilon}^{2-\varepsilon} t\,dt + \lim_{\delta \to +0} \int_{2+\delta}^{(5/2)+\delta} 2u\,du \\ &= \lim_{\varepsilon \to +0} \left[\frac{1}{2}t^2\right]_{(3/2)-\varepsilon}^{2-\varepsilon} + \lim_{\delta \to +0} \left[u^2\right]_{2+\delta}^{(5/2)+\delta} = \frac{25}{8}. \end{aligned}$$

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