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A normed spaces generated by a real sequence

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Abstract

We consider a fixed real convergent sequence $(a_n)_n$ and the set of all sequences $(x_n)_n$ for which the sequence $(x_n \cdot a_n)_n$ is also convergent. This paper presents the construction of a real normed spaces defined by this sequences.

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In the following we will denote by (c) the set of all convergent sequences and by (m) the set of all bounded sequences.

Let $(a_n)_n$ be a real sequence convergent to a.

Definition 1. *We say that a sequnce* $(x_n)_n$ *is* $(a_n)_n$*-convergent if the sequence* $(x_n \cdot a_n)_n$ *is convergent. We denote by* $(c(a_n)_n)$ *the set of all* $(a_n)_n$*-convergent sequences.*

Proposition 1 *The set* $(c(a_n)_n)$ *verifies the properties:*

1. $(c) \subset (c(a_n)_n)$
2. *If* $(a_n)_n \to a$ *with* $a \neq 0$ *then* $(c(a_n)_n) = (c)$.
3. *If* $(a_n)_n \to 0$ *then* $(m) \subset (c(a_n)_n)$.

Proof. 1. Indeed, if $(x_n)_n \in (c)$, then the sequence $(x_n \cdot a_n)_n \in (c)$.

2. Suppose that $(x_n)_n \in (c(a_n)_n)$. Than the sequence $(x_n \cdot a_n)_n$ is convergent. Because $(a_n)_n \to a$ and $a \neq 0$, there exists n_0 such that $a_n \neq 0$ for any $n > n_0$. For $n > n_0$ we have

$$x_n = (x_n \cdot a_n) \cdot \frac{1}{a_n}$$

and since the sequences

$$(\frac{1}{a_n})_{n>n_0}, (x_n \cdot a_n)_{n>n_0}$$

are convergent, it would follow that the sequence $(x_n)_{n>n_0}$ is convergent as well.

3. If $(x_n)_n \in (m)$ and $(a_n)_n \to 0$ the sequence $(x_n \cdot a_n)_n$ is obviously convergent to 0 and consequently, $(x_n)_n \in (c(a_n)_n)$.

Remark 1. *If* $a_n \to 0$ *there are divergent sequences belonging to the set* $(c(a_n)_n)$.

Indeed, if $a_n = \frac{1}{n^2}$, the sequence $(x_n)_n$ defined by $x_n = n$ belong to the set $(c(a_n)_n)$.

It results that $(c(a_n)_n)$ becomes very rich in elements when $(a_n)_n$ converges to 0. In this conditions, we will study the space $(c(a_n)_n)$ only for this case, obtaining some properties which characterize together the convergent sequences as well as the divergent ones.

Theorem 1. *The space $(c(a_n)_n)$ is a real linear space relative to the usual addition of sequences*

$$(x_n)_n + (y_n)_n = (x_n + y_n)_n,$$

respectively to the multiplication with a scalar

$$\lambda(x_n)_n = (\lambda x_n)_n.$$

Proof. If $(x_n)_n, (y_n)_n \in (c(a_n)_n)$ then $((x_n + y_n)a_n)_n = (x_n a_n)_n + (y_n a_n)_n$ is convergent and $(x_n + y_n)_n \in (c(a_n)_n)$. Similarly, if $(x_n)_n$ is a suite from $(c(a_n)_n)$ and $\lambda \in \mathbb{R}$ then $\lambda(x_n)_n = (\lambda x_n)_n$ is convergent and $\lambda(x_n)_n \in (c(a_n)_n)$.

Theorem 2. *The binary relation defined on $(c(a_n)_n)$ by $(x_n)_n \sim (y_n)_n$ if and only if*

$$\lim_{n\to\infty} x_n a_n = \lim_{n\to\infty} y_n a_n$$

is an equivalence relation.

The proof is immediately.

Theorem 3. *The quotient space $(c(a_n)_n)/\sim$, denoted by $(C(a_n)_n)$ is also a real linear space.*

Proof. We will show that the equivalence relation $\sim$ is compatible with the linear structure of $(c(a_n)_n)$.

Let $(x_n)_n \sim (x'_n)_n$ and $(y_n)_n \sim (y'_n)_n$. Then,

$$\lim_{n\to\infty}(x_n + y_n)a_n = \lim_{n\to\infty}(x_n a_n + y_n a_n) = \lim_{n\to\infty} x_n a_n + \lim_{n\to\infty} y_n a_n$$

and

$$\lim_{n\to\infty}(x'_n + y'_n)a_n = \lim_{n\to\infty}(x'_n a_n + y'_n a_n) = \lim_{n\to\infty} x'_n a_n + \lim_{n\to\infty} y'_n a_n.$$

But

$$\lim_{n\to\infty} x_n a_n = \lim_{n\to\infty} x'_n a_n \text{ and } \lim_{n\to\infty} y_n a_n = \lim_{n\to\infty} y'_n a_n$$

and, consequently, $(x_n + y_n)_n \sim (x'_n + y'_n)_n$.

Let $(x_n)_n \sim (x'_n)_n$ and $\lambda \in \mathbb{R}$. Then

$$\lim_{n\to\infty}(\lambda x_n)a_n = \lim_{n\to\infty} \lambda(x_n a_n) = \lambda \lim_{n\to\infty} x_n a_n =$$

$$= \lambda \lim_{n\to\infty} x'_n a_n = \lim_{n\to\infty} \lambda(x'_n a_n)) = \lim_{n\to\infty}(\lambda x'_n)a_n,$$

and, finally, $(\lambda x_n)_n \sim (\lambda x'_n)_n$.

To simplify the writing, we will denote in the following the elements of the space $(C(a_n)_n)$ by $\hat{x}, \hat{y}, \ldots.$

Theorem 4. *The real valued function defined on* $(C(a_n)_n)$ *by*

$$\hat{x} \mapsto \|\hat{x}\| = \lim_{n\to\infty} |x_n a_n|,$$

where $(x_n)_n \in \hat{x}$*, is a norm on* $(C(a_n)_n)$*. Thus,* $(C(a_n)_n)$ *becomes a normed linear space.*

Proof. First, let us show that the value of the function is independent of the particular representative chosen in the class $\hat{x}$ (the function is well-defined). Let

$$(x'_n)_n,\ (x''_n)_n \in \hat{x},$$

hence

$$\lim_{n\to\infty} x'_n a_n = \lim_{n\to\infty} x''_n a_n.$$

It results that the sequences $(|x'_n a_n|)_n,\ (|x''_n a_n|)_n$ are also convergent and have the same limit.

We have

$$\|\hat{0}\| = \lim_{n\to\infty} |0a_n| = 0$$

and conversely, if

$$\|\hat{x}\| = 0, \text{ then} \lim_{n\to\infty} |x_n a_n| = 0,$$

where $(x_n)_n \in \hat{x}$. Then also

$$\lim_{n\to\infty} x_n a_n = 0,$$

therefore $(x_n)_n \in \hat{0}$, this means $\hat{x} = \hat{0}$.

Let $\alpha \in \mathbb{R}$ and $\hat{x} \in (C(a_n)_n)$. If $(x_n)_n \in \hat{x}$ then $(\alpha x_n)_n \in \alpha\hat{x}$ and we can write

$$\|\alpha\hat{x}\| = \lim_{n\to\infty} |(\alpha x_n)a_n| = |\alpha| \lim_{n\to\infty} |x_n a_n| = |\alpha|\|\hat{x}\|.$$

Let $\hat{x}, \hat{y} \in (C(a_n)_n)$ and $(x_n)_n \in \hat{x},\ (y_n)_n \in \hat{y}$. Obviously,

$$(x_n + y_n)_n \in \hat{x} + \hat{y}$$

and, consequently,

$$\|\hat{x}+\hat{y}\| = \lim_{n\to\infty} |(x_n+y_n)a_n| = \lim_{n\to\infty} |x_n a_n + y_n a_n| \leq \lim_{n\to\infty} (|x_n a_n| + |y_n a_n|) =$$

$$\lim_{n\to\infty} |x_n a_n| + \lim_{n\to\infty} |y_n a_n| = \|\hat{x}\| + \|\hat{y}\|.$$

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